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The Blocks of a Semisimple Algebraic Group*

STEPHEN DONKIN

Department of Mathematics, University of Warwick, Coventry, Warks CV4 7AL, England

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Let G be an affine, connected, semisimple, simply connected algebraic group over an algebraically closed field of prime characteristic p. In this paper we give an exact description of the blocks of the rational representation of G. We show that, if the root system of G is indecomposable, the block $\mathscr{B}(\lambda)$ containing a dominant weight λ is precisely the set of dominant weights which are conjugate to λ under the "dot" action of the affine Weyl group $W_{p^{r(\lambda)+1}}$ *(r(A)* is defined in Section 2) of G. From this one may easily obtain the blocks of an arbitrary semisimple group. The result was obtained by Winter for the two dimensional special linear group in [15] and, for G of arbitrary type and λ "p-regular," by Humphreys and Jantzen in Section 2.4 of [10]. The analogous result for the hyperalgebra of an infinitesimal subgroup of G, for general λ, was proved by Jantzen in Section 5.5 of [11].

There are two parts to this problem. On the one hand we must show that the set of dominant weights which are $W_p r(\lambda) + 1$ conjugate to λ is a union of blocks and, on the other, we must show that this set is contained in a block. The first part of our solution to this problem builds on recent work of Andersen [I], which implies the desired conclusion when $r(\lambda)$ is zero. The appropriate result here is essentially proved in Section 2.3 of [10], though our proof is based on that in the author's thesis (Section 4(A) of [4]). This is dealt with, after the preliminaries of Section 1, in Section 2. Sections 3 to 5 contain the second part of our solution to this problem. In Section 3 we show that any block must contain an element arbitrarily distant from each wall of the dominant region. In Section 4 we show that if λ is any weight, not equal to minus half the sum of the positive roots, and a any simple root then one can go from λ to $\lambda - p^{r(\lambda)+1}a$ by a finite number, bounded independently of

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λ, of "elementary moves." The proof here is obtained by a detailed examination of the proof of (3) of 5.5 of [11]. This is used to show that if τ is a dominant weight, sufficiently far from the walls of the dominant region. then τ and $\tau - p^{r(\tau)+1}\alpha$ are in the same block. When coupled with the work in Section 3 this yields, in Section 5, the desired description of the blocks.

1. PRELIMINARIES

To fix ideas we take G to be a universal Chevalley group constructed as in [13]. Let g be a finite dimensional, complex, simple Lie algebra, h a Cartan subalgebra and R the root system. We choose a Chevalley basis $\{X_\alpha, H_i \; 1 \leqslant i \leqslant l, \alpha \in R\}$ of g and let $\mathbf{U}_{\mathbb{Z}}$ denote the $\mathbb{Z}$ subalgebra of the universal enveloping algebra of g generated by all $X_\alpha^r/r!$ with $\alpha \in R$ and $r \geqslant 1$. Let V be a finite dimensional g module, affording the representation π, such that the weights of V span the lattice of integral weights of h. Now we take G to be the Chevalley group determined by π, an admissible $\mathbb{Z}$ form V of V and an algebraically closed field K of prime characteristic p.

Let $A = K[G]$, the coordinate ring of G. Identifying $A \otimes_K A$ with a subalgebra of the K-algebra of K-valued functions on $G \times G$, in the obvious manner, we obtain a K-algebra map $\mu: A \to A \otimes A$, satisfying $\mu(f)(x, y) = f(xy)$ for all $f \in A$, and $x, y \in G$. Thus we have a Hopf algebra (A, μ, ε) over K, where $\varepsilon: A \to K$ is defined by $\varepsilon(f) = f(1)$ for $f \in A$. The antipode S of A satisfies $S(f)(x) = f(x^{-1})$ for $f \in A$, $x \in G$. The coalgebra structure on A gives rise, naturally, to an algebra structure on $\Gamma = \mathrm{Hom}_K(A, K)$. multiplication being given by

$$(\gamma_1 * \gamma_2)(f) = (\gamma_1 \otimes \gamma_2)\,\mu(f)$$

for $\gamma_1, \gamma_2 \in \Gamma$, $f \in A$.

For each root α there is a morphism of algebraic groups φ_α, from the additive group of the field K to G, given by $\varphi_\alpha(t) = x_\alpha(t)$, where $x_\alpha(t)$ is the generator of G defined in Section 3 of [12]. It follows that there are uniquely determined elements $\xi_{\alpha,r}$ of Γ such that, for any f in A, $\xi_{\alpha,r}(f)$ is zero for all but a finite number of r and

$$f(x_\alpha(t)) = \sum_{r=0}^{\infty} t^r \xi_{\alpha,r}(f)$$

for all t in K.

For any root α and positive integer r we write $X_{\alpha,r}$ for the element $X_\alpha^r/r! \otimes 1$ of $\mathbf{U}_K$, where $\mathbf{U}_K = \mathbf{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. By convention $X_{\alpha,0}$ is the identity element of $\mathbf{U}_K$. We know, from Sections 6.5, 6.6 and 9.1 of [3], that there is a monomorphism of K-algebras $\psi: \mathbf{U}_K \to \Gamma$ satisfying $\psi(X_{\alpha,r}) = \xi_{\alpha,r}$ for all roots α, and positive integers r. Moreover, we also know from [3] that $\psi(\mathbf{u}_n)$ is the subalgebra of Γ consisting of those elements which vanish on $\mathscr{M}^{[p^n]}$, the ideal of A generated by $\{f^{p^n} : f(1) = 0\}$. Here, for a positive integer n, $\mathbf{u}_n$ denotes the subalgebra of $\mathbf{U}_K$ generated by $\{X_{\alpha,r} : \alpha \in R, 1 \leqslant r < p^n\}$.

Any rational G module V may be regarded as a locally finite (dimensional) $\mathbf{U}_K$ module by defining

$$uv_i = \sum_{j\in I} \psi(u)(f_{ji})\, v_j$$

for any $u \in \mathbf{U}_K$, $i \in I$ and extending this action K-linearly to the whole of V. In the above $\{v_i : i \in I\}$ is a basis of V and the elements f_{ij} of A are determined by the equations

$$gv_i = \sum_{j\in I} f_{ji}(g)\, v_j \qquad (g \in G).$$

Conversely, given any locally finite $\mathbf{U}_K$ module V, Cline, Parshall and Scott have shown, in Sections 6.8 and 9.2 of [3] (see also [14]), that V may naturally be regarded as a rational G module with action given on the generators $x_\alpha(t)$ ($\alpha \in R$, $t \in K$) by

$$x_\alpha(t)v = \sum_{r=0} t^r X_{\alpha,r} v$$

for $v \in V$ (it turns out that all but a finite number of the $X_{\alpha,r}v$ are zero). This transfer of action determines an inclusion preserving equivalence of categories from locally finite $\mathbf{U}_K$ modules to rational G modules.

Throughout this paper, T denotes the standard maximal torus arising from the Chevalley construction of G and W denotes the Weyl group. The weight lattice of T is denoted by X, the set of dominant weights by X^+ and (,) denotes the positive definite, W invariant, bilinear, symmetric, nonsingular form on X obtained from the Killing form on h.

2. An Operator on the Blocks

We progress by means of a general proposition on the representation theory of Hopf algebras. For a morphism $\theta: A \to B$ of commutative Hopf algebras we denote by θ_0 (respectively θ^0) the θ-restriction (respectively θ-induction) functor from the category of (right) A comodules (respectively B comodules) to the category of B comodules (respectively A comodules). These functors are discussed at length in Section 3 of [5].

Recall that the socle of a module (respectively comodule) is the sume of its simple submodules (respectively subcomodules).

PROPOSITION 2.1. *Let (H, μ, ε) be a commutative Hopf algebra over an algebraically closed field K, $\mathscr{M}$ = ker ε, A a sub-Hopf algebra of H and $J = (A \cap \mathscr{M})H$. Let $\pi: H \to H/J$ and $j: A \to H$ be the natural maps. Suppose that V is an H comodule such that $\pi_0(V)$ is simple and W is an A comodule with a simple socle. Then $V \otimes j_0(W)$ has a simple socle. Furthermore, if W is simple, $V \otimes j_0(W)$ is simple.*

Proof. (Based on an idea of H. Blau). Let $\{W_\tau : \tau \in \mathscr{S}\}$ be a full set of simple A comodules. Let m_τ be the K-dimension of W_τ and E_τ be the rationally injective envelope of W_τ for each τ in $\mathscr{S}$. Suppose that the socle of W is isomorphic to W_{τ_1}.

Now let Q be any finite dimensional H comodule such that $\pi_0(Q)$ is a direct sum of copies of n,(V). It follows from Schur's lemma that

$$\dim \mathrm{Hom}_{H/J}(\pi_0(Q), \pi_0(V)) = \dim Q/\dim V. \tag{1}$$

However, the reciprocity law and tensor identity for induction (3(c) and 3(h) of [5]) give

$$\begin{aligned}\dim \mathrm{Hom}_{H/J}(\pi_0(Q), \pi_0(V)) &= \dim \mathrm{Hom}_H(Q, \pi^0(\pi_0(V))) \\ &= \dim \mathrm{Hom}_H(Q, V \otimes \pi_0(K)).\end{aligned} \tag{2}$$

By Corollary 4.2 of [5], $\pi^0(K) \cong j,(A)$ and by Green's structure theorem for coalgebras (1.5g of [6])

$$A \cong \sum_{\tau \in \mathscr{S}} \oplus \sum_{i=1}^{m_\tau} \oplus E_\tau \tag{3}$$

as right A comodules. Combining (1), (2) and (3) we obtain

$$\dim Q/\dim V = \sum_{\tau \in \mathscr{S}} m_\tau \dim \mathrm{Hom}_H(Q, V \otimes j_0(E_\tau)). \tag{4}$$

We first consider the case in which W is simple; that is, $W = W_{\tau_1}$. Let Q be a nonzero H submodule of $V \otimes j_0(W)$. The inclusion map from Q to $V \otimes j_0(W)$ gives a nonzero element of $\mathrm{Hom}_H(Q, V \otimes j_0(W_{\tau_1}))$ and from (4) we obtain

$$\dim Q/\dim V \geqslant m_{\tau_1} = \dim W.$$

Thus $\dim Q \geqslant \dim V \otimes W$. The reverse inequality holds since Q is a subspace of $V \otimes W$. Hence $Q = V \otimes W$ and $V \otimes j_0(W)$ is a simple H comodule.

We now consider the general case. We put $Q_{\tau'} = V \otimes j_0(W_{\tau'})$ for any τ' in $\mathscr{S}$. Applying (4), with $Q = Q_{\tau'}$, we obtain

$$\dim \operatorname{Hom}_H(Q_{\tau'}, V \otimes j_0(E_\tau)) = \delta_{\tau,\tau'},$$

the Kronecker delta, for any $\tau, \tau' \in \mathscr{S}$. In particular

$$\dim \operatorname{Hom}_H(Q_\tau, V \otimes j_0(E_{\tau_1})) = \delta_{\tau,\tau_1}. \tag{5}$$

Now if P is any simple H comodule which occurs in the socle of $V \otimes j_0(E_{\tau_1})$ then P is a composition factor of $V \otimes j_0(W_\tau)$ for some τ in $\mathscr{S}$ and so, by the case already considered, P is isomorphic to Q_τ. Using Schur's lemma, one can see that the number of times that Q_τ occurs in the socle of $V \otimes j_0(E_{\tau_1})$ is recorded by the dimension of $\operatorname{Hom}_H(Q_\tau, V \otimes j_0(E_{\tau_1}))$. It now follows from (5) that the socle of $V \otimes j_0(E_{\tau_1})$ is isomorphic to the simple comodule Q_{τ_1}. Since the socle of W is isomorphic to W_{τ_1} we may identify W with a subcomodule of E_{τ_1} and so $V \otimes j_0(W)$ with a subcomodule of $V \otimes j_0(E_{\tau_1})$. Hence $V \otimes j_0(W)$ has a simple socle, as required.

Let $H = K[G]$, the coordinate ring of our universal Chevalley group over the algebraically closed field K of prime characteristic p. Any rational left G module V with basis $\{v_i : i \in I\}$ naturally gives rise to a right H comodule $\mathscr{F}(V) = (V, \tau)$, where the K-map $\tau: V \to V \otimes_K H$ satisfies

$$\tau(v_i) = \sum_{j \in I} v_j \otimes f_{ji}$$

for any $i \in I$. The elements f_{ij} of H are determined by the equations

$$gv_i = \sum_{j \in I} f_{ji}(g)\, v_j \qquad (i \in I, g \in G).$$

Moreover, $\mathscr{F}$ determines an equivalence of categories between rational left G modules and right H comodules.

For a dominant weight λ we denote by $L(\lambda)$ the simple rational G module with highest weight λ. The simple module with highest weight $(p^n - 1)\rho$, where ρ is half the sum of the positive roots, is called the nth Steinberg module and may also be denoted by St_n. When $n = 1$ we may simply write St. The rationally injective envelope of $L(\lambda)$ will be denoted by $I(\lambda)$ and Fr will denote the Frobenius morphism on rational G modules.

We now take, in Proposition 2.1, $A = H^{(p)}$, the sub-Hopf algebra of H consisting of pth powers of elements of $H = K[G]$. Thus $J = H(A \cap \mathscr{M}) = \mathscr{M}^{[p]}$. By Section 5.5 of [8] St is projective as a $\mathbf{u}_1$ module. However, $\mathbf{u}_1$ is a finite dimensional Hopf algebra and therefore Frobenius (in fact, by [9], each algebra $\mathbf{u}_n$ is symmetric). Thus St is injective as a $\mathbf{u}_1$ module. It is well known that St is simple as a $\mathbf{u}_1$ module and it follows that St is simple and injective as an $H/\mathscr{M}^{[p]}$ comodule. Hence we obtain from Proposition 2.1, together with Corollary 4.4 of [5],

COROLLARY 2.2. *For any dominant weight* λ,

$$St \otimes I(\lambda)^{Fr} \cong I((p \quad 1)\rho + p\lambda)$$

as a G module.

Remarks 1. Proposition 2.1 may also be used as the basis of a proof of Steinberg's twisted tensor product theorem. This is spelled out in Section 2.4(A) of [4].

2. Readers not as fond of Hopf algebras as the author may obtain a proof of Corollary 2.2 which is independent of Propositon 2.1 by using the formulas 2.3(3) and 2.3(1) of [12].

Let λ and λ' be dominant weights. It follows from 2.5.5 of [6] that $\mathrm{Hom}_G(I(\lambda), I(\lambda'))$ is nonzero if and only if $\mathrm{Hom}_G(I(\lambda')$, Z(A)) is nonzero. We say that λ and λ' are adjacent if $\mathrm{Hom}_G(I(\lambda), I(\lambda'))$ is nonzero and let $\sim$ be the equivalence relation generated by adjacency. We think of a block of G as an equivalence class of dominant weights with respect to this relation (see 1.6 of [6] for a general discussion of blocks of comodules). It is not difficult to show that if $L(\lambda)$ and L(X) are composition factors of some indecomposable G module then λ and λ' are in the same block.

We define an operator θ: $X^+ \to X^+$ on the dominant weights by $\theta(\lambda) =$ (p $- 1)\rho + p\lambda$, for any λ in X^+.

COROLLARY 2.3. *If* $\mathscr{B}$ *is a block then* $\theta(\mathscr{B})$ *is a block.*

Proof. Suppose that λ and τ are adjacent dominant weights. Thus $L(\lambda)$ is a composition factor of Z(r) and so $L((p-1)\rho + p\lambda)$, by Steinberg's twisted tensor product theorem isomorphic to $St \otimes L(\lambda)^{Fr}$, is a composition factor of $St \otimes$ Z(t)“'. According to Corollary 2.2 this implies that $L((p-1)\rho + p\lambda)$ is a composition factor of $I((p - 1)\rho + p\tau)$; that is, $(p - 1)\rho + p\lambda$ and $(p - 1)\rho + p\tau$ are adjacent. Since $\sim$ is the equivalence relation generated by adjacency this shows that any two elements of $\theta(\mathscr{B})$ are equivalent.

To complete the proof we must show that if η is adjacent to $\theta(\lambda)$, for some element λ of $\mathscr{B}$ then η lies in $\theta(\mathscr{B})$. In this case $L(\eta)$ is a composition factor of $I(\theta(\lambda))$ which, by Corollary 2.2, is isomorphic to St $\otimes I(\lambda)^{Fr}$. It is not difficult, using Steinberg's tensor product theorem, to show that any composition factor of St $\otimes I(\lambda)^{Fr}$ has the form St $\otimes L(\tau)^{Fr}$, where $L(\tau)$ is a composition factor of Z(n). In particular $\eta = \theta(\tau)$ for some 7 $\in \mathscr{B}$ and we are done.

Let λ be a dominant weight. We denote by $V(\lambda)$ the simple, finite dimensional module, of highest weight λ, for the simple Lie algebra g. Let v be a nonzero element in the λ weight space of $V(\lambda)$, the finite dimensional $\mathbf{U}_K$ module (and hence rational G module) $W(\lambda) = U_{\mathbb{Z}} v \otimes_{\mathbb{Z}} K$ is called the Weyi module of highest weight λ. The character of $W(\lambda)$ is given by Weyl's character formula.

For any integral weight I, not equal to $-\rho$, we define $r(\lambda)$ to be the nonnegative integer satisfying

$$\lambda + \rho \in p^{r(\lambda)}X \backslash p^{r(\lambda)+1}X.$$

The "dot" action of W on X is given by

$$w \cdot \mu = w(\mu + \rho) - \rho$$

for $w \in W, \mu \in X$. We denote by $\mathbb{Z}R$ the root lattice of G.

Corollary 2.4. *For any dominant weight λ,*

$$(W \cdot \lambda + p^{r(\lambda)+1}\mathbb{Z}R) \cap X^+$$

is a union of blocks.

Proof. We first consider the case in which $r(\lambda) = 0$. It may be deduced (see Section 2.3 of [10]) from Green's version (2.5.4 of [6]) of Brauer's reciprocity law that it is enough to check that if τ, η are dominant, 7 is an element of $W\lambda + p\mathbb{Z}R$ and $L(\eta)$ is a composition factor of *W(z),* then η is also an element of $W\lambda + p\mathbb{Z}R$. However, this is an easy consequence of the strong linkage principle, recently proved by Andersen (see Corollary 3 of [1]).

Now suppose that λ is any dominant weight. We may write $\lambda = \theta^{r(\lambda)}(\lambda')$ for some dominant I' with $r(\lambda') = 0$. By the above $(W\lambda' + p\mathbb{Z}R) \cap X^+$ is a union of blocks and so, by Corollary 2.3,

$$\theta^{r(\lambda)}(W \cdot \lambda' + p\mathbb{Z}R) \cap X^+ = (W \cdot \lambda + p^{r(\lambda)+1}\mathbb{Z}R) \cap X^+$$

is a union of blocks, as required.

3. Blocks Intersect Certain Regions

For any real number A we define

$$X(A) = \{\lambda \in X \mid (\lambda, \alpha^{\vee}) \geqslant A \text{ for each } \alpha \in B\},$$

where B denotes the set of simple roots and a" the coroot $2\alpha/(\alpha,$ a) associated with a. In this section we show that X(A) contains an element of each block. This allows us, in finding the blocks, to go from a dominant weight λ, close to the walls of the dominant region, to a weight in the same block as λ, in the hinterland. We could not escape from such awkward corners using oly the elementary moves of Section 4.

For a positive integer *m we* put

$$X_m = \{\lambda \in X^+ \mid (\lambda, \alpha^\vee) < p^m \text{ for each } \alpha \in B\}.$$

In the proof of the following lemma we work in Jantzen's category of $\mathbf{u}_m - T$ modules. As in [1 1], $\hat{L}(m, \lambda)$ denotes the simple $\mathbf{u}_m - T$ module having highest weight λ, $\hat{Q}(m, \lambda)$ the projective envelope of $\hat{L}(m, \lambda)$ as a $\mathbf{u}_m - T$ module and $\hat{Z}(m, \lambda)$ the $\mathbf{u}_m - T$ module induced from the one dimensional $\mathbf{b}_m - T$ module of weight λ. Here $\mathbf{b}_m$ denotes the subalgebra of $\mathbf{u}_m$ corresponding to the Bore1 subgroup of G generated by all x,(t) (a > 0, $t \in$ K).

LEMMA 3.1. ***Suppose*** $\lambda \in X^+$, $\tau \in B$ ***and***

$$(\lambda + \rho, a") = bp' + ap^{r+1}$$

for integers a, b and r with 0 $< b < p$***. Then, for each m*** $> r$ ***with*** $\lambda \in X$,, ***there is some*** $\tau_m \in X$ ***such that***

$$\lambda - bp^r\alpha + p^m\tau_m \in \mathscr{B}(\lambda),$$

the block containing λ.

Proof. By Section 5.5 of [11] there is a nonzero homomorphism of $\mathbf{u}_m - T$ modules from $\hat{Z}(m, \lambda - bp^r\alpha)$ to $\hat{Z}(m, \lambda)$. Hence $\hat{L}(m, \lambda - bp^r\alpha)$ is a $\mathbf{u}_m - T$ composition factor of $\hat{Z}(m, \lambda)$. As $\hat{Z}(m, \lambda)/\text{rad}\, \hat{Z}(m, \lambda)$ is isomorphic to $\hat{L}(m, \lambda)$ and as $\hat{Q}(m, \lambda)$ is defined as the projective cover of $\hat{L}(m, \lambda))$, there is a surjective homomorphism $\hat{Q}(m, A.) \to \hat{Z}(m, A)$, hence $\hat{L}(m, \lambda) - bp'a)$ is a composition factor of $\hat{Q}(m, A)$.

Let $\lambda_m^0 = (p" - 1)\rho + w_0\lambda$, where w_0 is the longest element of W. We know, from the proof of Lemma 4 of [2], that $L(\lambda)$ occurs exactly once as a $\mathbf{u}_m$ submodule and exactly once as a G submodule of $L(\lambda_m^0) \otimes$ St,. We let $U_m(\lambda)$ be the indecomposable G component of $L(\lambda_m^0) \otimes$ St, containing the copy of $L(\lambda)$. Since St, is projective as a $\mathbf{u}_m - T$ module so is $L(\lambda_m^0) \otimes$ *St,,,* also and it follows that $U_m(\lambda)$ is projective as a $\mathbf{u}_m - T$ module. Thus $\hat{Q}(m, \lambda)$ is a $\mathbf{u}_m - T$ component of $U_m(\lambda)$ and so $\hat{L}(m,$ A $- bp^r\alpha)$ is a $\mathbf{u}_m - T$ composition factor of $U_m(\lambda)$. Hence there is a G composition factor $L(\eta)$ of $U_m(\lambda)$ such that $\hat{L}(m, \lambda - bp'a)$ is a $\mathbf{u}_m - T$ composition factor of $L(\eta)$. By Steinberg's twisted tensor product theorem, $L(\eta)$ is isomorphic to $L(\eta_1) \otimes L(\eta_2)^{Fr^m}$, where $\eta = \eta_1 + p^{m\eta_2}$ and η_1 belongs to X_m.

It follows from (2) of Section 2.8 of [11] that

$$\lambda - bp'a = \eta_1 + p^m\xi$$

for some weight ξ of $L(\eta_2)$. Thus we have

$$\eta = \lambda - bp^r\alpha + p^m\tau_m,$$

where $\tau_m = \eta_2 - \xi$.

Now λ and η are in the same block since $L(\lambda)$ and $L(\eta)$ are composition factors of the indecomposable G module $U_m(\lambda)$. This proves the lemma.

PROPOSITION 3.2. ***For any real number A and any block $\mathscr{B}$, $\mathscr{B} \cap X(A)$ is not empty.***

Proof. Suppose, for a contradiction, that $\mathscr{B} \cap X(A)$ is empty. For any λ in $\mathscr{B}$ we define

$$S(\lambda) = \{a \in B \mid (\lambda, a") < A\}.$$

Of course, by the supposition, for each λ in $\mathscr{B}$, S(A) is not empty. Let

$$s = \text{minimum}\{|S(\lambda)| : \lambda \in \mathscr{B}\}$$

and

$$\mathscr{B}_s = \{\lambda \in \mathscr{B} : |S(\lambda)| = s\}.$$

We choose **first** some a $\in B$ such that $(\lambda$, a") $< A$ for some $\lambda \in \mathscr{B}_s$ and then $\mu \in \mathscr{B}_s$ such that $(\mu$, a") is as small as possible. By the lemma, for all m **sufficiently** large, there is some $\tau_m \in$ X such that

$$\mu_m = \mu - bp^r\alpha + p^m\tau_m \in \mathscr{B},$$

where $(\mu + p, a") = bp' + ap^{r+1}$ for some $0 < b < p$.

Now if $\beta \in S(\mu_m)$, $\beta \neq a$ *we* have

$$(\mu_m, \beta^\vee) = (\mu, \beta^\vee) - bp^r(\alpha, \beta^\vee) + p^m(\tau_m, \beta^\vee)$$

with $(\alpha, \beta^\vee) \leqslant 0$. But μ_m is dominant so $(\mu_m, \beta^\vee) \geqslant 0$ and therefore $(\tau_m, \beta^\vee) \geqslant 0$ **for** all m large. However, $(\mu_m, \beta^\vee) < A$ and so it follows that $(\tau_m, \beta^\vee) \leqslant 0$ for all m large. Thus we obtain $(\tau_m, \beta^\vee) = 0$ for all m large and so

$$(\mu_m, \beta^\vee) = (\mu, \beta^\vee) - bp^r(\alpha, \beta^\vee) \geqslant (\mu, \beta^\vee),$$

which implies that $\beta \in S(\mu)$. We have shown that, for m large, if $\beta \in S(\mu_m)$ and $\beta \neq \alpha$ then $\beta \in S(\mu)$. Since a $\in S(\mu)$, by the choice of μ, we have $S(\mu_m) \subseteq S(\mu)$. Now, by the minimality of $|S(\mu)|$, we must have $S(\mu_m) = S(\mu)$ and so a $\in S(\mu_m)$, for all m large. We obtain, from the definition of $S(\mu_m)$

$$(\mu_m, \alpha^\vee) = (\mu, a") - 2bp^r + p^m(\tau_m, a') < A.$$

As above, we obtain $(\tau_m$, a") = 0, for all m large, and so $(\mu_m$, a") $< (\mu$, a"), contradicting the minimality of $(\mu$, a"). This contradiction proves the proposition.

4. Sequences of Elementary Moves

For a weight λ and a simple root a we define $r(\lambda$, a) $= \infty$ if $(\lambda + \rho$, a") = 0 and define $r(\lambda, a)$ by $(\lambda + \rho, a") \in p^{r(\lambda,\alpha)}\mathbb{Z} \backslash p^{r(\lambda,\alpha)+1}\mathbb{Z}$ otherwise. Thus we have

$$r(\lambda) = \text{minimum}\{\ r(\lambda, a)\quad a \in B\ \}$$

if λ is not equal to $-\rho$. We put $r(-\rho) = \infty$.

For a nonnegative integer r we define (more or less as in Section 2.4 of [10]) $X^{(r)} = \{\lambda \in X \mid r(\lambda) = r\}$. We put $X^{(\infty)} = \{-\rho\}$.

For any $\lambda \in X$ and $a \in B$ such that $(\lambda + \rho, a") \neq 0$ we defme integers $b(\lambda, \alpha), a(\lambda$, a) by

$$(\lambda + p, a") = b(\lambda, \alpha)\, p^{r(\lambda,\alpha)} + a(\lambda, a)\ p^{r(\lambda,\alpha)+1}$$

with $0 < b(\lambda, a) < p$.

For $\lambda, \mu \in X$ we shall write $\lambda \to \mu$ if $\mu = \lambda - b(\lambda, a)\ p^{r(\lambda,\alpha)}$ for some a $\in B$ with $r(\lambda$, a) equal to $r(\lambda)$ or $r(\lambda) + 1$. We write, for $\lambda, \mu \in X, d(\lambda, \mu) = n$ if n is the length of a shortest sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_n = \mu$ such that, for each $0 \leqslant i < n$, $\lambda_i \to \lambda_{i+1}$ o r $\lambda_{i+1} \to \lambda_i$. If no such sequence exists we write $d(\lambda, \mu)$ = co. We say that λ and μ are related by an elementary move if $d(\lambda, \mu) = 1$ and say that λ and μ are related by a sequence of elementary moves if $d(\lambda, \mu)$ is finite. We shall see, in Section 5, that if λ and μ are dominant weights, related by a sequence of elementary moves and $d(\lambda, \mu)$ is small compared to the distance of λ and μ from the walls of the dominant chamber, then they are in the same block.

Let A be a weight not equal to $-\rho$. For our final technical definition we put $m(\lambda$, a) = m if m is the shortest length of a sequence $a = a,, a,,\ldots, a$, *of* simple roots such that $(\alpha_{i+1}, \alpha_i^\vee) < 0$ for $i = 1, 2, \ldots$, m $-$ 1 and $r(\lambda, a,) = r(\lambda)$.

The proof of the following proposition was obtained by counting the number of elementary moves needed in the proof of (3) of Section 5.5 of [11].

PROPOSITION 4.1. *For any $\lambda \in X^{(r)}$, $\alpha \in B$ and positive integer k*

$$d(\lambda, \lambda - kp^{r+1}\alpha) < k(2p)^{m(\lambda,\alpha)}.$$

Proof: The proof is by induction on $m = m(\lambda, \alpha)$. We suppose first that $m(\lambda, \alpha) = 1$. Then $(\lambda + \rho, \alpha^\vee) \equiv bp^r \bmod p^{r+1}$ for some $0 < b < p$. Thus

$$d(\lambda, \lambda - bp^r\alpha) = 1. \tag{1}$$

Now $(\lambda - bp^r\alpha + \rho, \alpha^\vee) \equiv bp^r - 2bp^r \equiv (p-b)p^r \bmod p^{r+1}$. Hence $d(\lambda - bp^r\alpha, \lambda - bp^r\alpha - (p-b)\,p^r\alpha) = 1$; that is,

$$d(\lambda - bp^r\alpha, \lambda - p^{r+1}\alpha) = 1. \tag{2}$$

Now (1) and (2) give

$$d(\lambda, \lambda - p^{r+1}\alpha) \leqslant 2 < 2p \tag{3}$$

using the triangle inequality for $d(\ ,\ )$. Now repeated application of (3) gives

$$d(\lambda, \lambda - kp^{r+1}\alpha) < 2kp$$

for any positive integer k, as required.

We now assume that $m > 1$ and that the proposition holds for $\lambda' \in X^{(r)}$, $\beta \in B$ with $m(\lambda', \beta) < m$. Clearly it suffices to prove the proposition with $k = 1$.

Case (i). *p does not divide $(\alpha_m, \alpha^\vee_{m-1})$*

Now $(\lambda + \rho, \alpha^\vee_m) = bp^r + ap^{r+1}$, where $0 < b < p$. It is easy to see that $\lambda \to \lambda - bp^r\alpha_m$ and $\lambda - p^{r+1}\alpha \to \lambda - p^{r+1}\alpha - bp^r\alpha_m$. Hence, by the triangle inequality,

$$d(\lambda, \lambda - p^{r+1}\alpha) \leqslant 2 + d(\lambda - bp^r\alpha_m, \lambda - p^{r+1}\alpha - bp^r\alpha_m). \tag{4}$$

Now putting $\lambda' = \lambda - bp^r\alpha_m$ we have $\lambda' \in X^{(r)}$ and $(\lambda' + \rho, \alpha^\vee_{m-1}) = (\lambda + \rho, \alpha^\vee_{m-1}) - bp^r(\alpha_m, \alpha^\vee_{m-1})$. By the minimality of m, $(\lambda + \rho, \alpha^\vee_{m-1}) \equiv 0 \bmod p^{r+1}$; hence $(\lambda' + \rho, \alpha^\vee_{m-1}) = -bp^r(\alpha_m, \alpha^\vee_{m-1}) \not\equiv 0 \bmod p^{r+1}$. Thus $m(\lambda', \alpha) < m(\lambda, \alpha)$ and (4) together with the inductive assumption gives

$$d(\lambda, \lambda - p^{r+1}\alpha) < 2 + (2p)^{m-1} < (2p)^m$$

as required.

Case (ii). $(\alpha_2, \alpha^\nu) = -1$

It is evident, from the minimality of ***m***, that $m(\lambda, a,) = m - 1$. Hence the inductive assumption gives, for any $n \geqslant 0$,

$$d(\lambda - np^{r+1}\alpha_2, \lambda) \leqslant n(2p)^{m-1} \tag{5}$$

and

$$d(\lambda - p^{r+1}\alpha, \lambda - p^{r+1}\alpha - np^{r+1}\alpha_2) \leqslant n(2p)^{m-1}. \tag{6}$$

Now $(A - np^{r+1}\alpha_2 + \rho, \text{a''}) = (\lambda + \rho, \text{a'}) + np^{r+1}$, so that, for some $0 \leqslant n_0 < p$, ***we*** have $(\lambda - n_0 p^{r+1}\alpha_2 + \rho, \text{a''}) \equiv p^{r+1} \bmod p^{r+2}$. Thus

$$\lambda - n_0 p^{r+1} a, \rightarrow \lambda - n_0 p^{r+1}\alpha_2 - p^{r+}\text{'}a. \tag{7}$$

Now using (5) and (6) with $n = n_0$ we obtain

$$\begin{aligned} &d(\lambda, \lambda - p^{r+}\text{'}a) \\ &\qquad \leqslant 2n_0(2p)^{m-1} + d(\lambda - n_0 p^{r+1}\alpha_2, \lambda - n_0 p^{r+1}\alpha_2 - p^{r+}\text{'}a) \end{aligned}$$

by the triangle inequality. Using (7) this becomes

$$d(\lambda, \lambda - p^{r+1}\alpha) \leqslant 2n_0(2p)^{m-1} + 1,$$

which is less than $(2p)^m$, thus completing Case (ii).

Since the Dynkin diagram of an indecomposable root system cannot have more than one multiple bond we must have ***m*** = 2 and $(\alpha_2, a'') = -p$ in the cases not covered by (i) and (ii).

Case (iii) ***m = 2, $(\alpha_2, a') = -2$, p = 2***

In this case the subroot system generated by ***a*** and ***a,*** is of type B_2 and so (a, α_2^ν) = -1.

If $(\lambda + \rho, a'') \equiv 2^{r+1} \bmod 2^{r+2}$ then $\lambda \rightarrow \lambda - 2^{r+1}\alpha$ so

$$d(\lambda, \lambda - 2^{r+1}\alpha) = 1 < (2p)^2 = 16$$

as required.

On the other hand, if $(\lambda + \text{p}, a'') \equiv 0 \bmod 2^{r+2}$ then $(\lambda + \rho - 2^r\alpha_2, \text{a''}) = (\lambda + \rho, al') + 2^{r+1} \equiv 2^{r+1} \bmod 2^{r+2}$ so we obtain

$$\lambda \rightarrow \lambda - 2^r\alpha_2 \rightarrow \lambda - 2^r\alpha_2 - 2^{r+1}\alpha.$$

We also have $\lambda - 2^{r+1}\alpha \rightarrow \lambda - 2^r\alpha_2 - 2^{r+}$'a, which proves

$$d(\lambda, \lambda - 2^{r+1}\alpha) \leqslant 3 < 16 = (2p)^2$$

as required.

Case (iv). ***m* = 2, (*a*,, *a*”) = -3,** p **= 3.**

In this case ***R*** must be of type G_2 and (a, a;‘) = -1. Now (λ + ***p*, *al*’)** $\equiv$ $3^{r+1}c$ mod 3^{r+2} for some $0 \leqslant$ c $\leqslant 2$. If c = 1 then $\lambda \to \lambda - 3^{r+1}\alpha$ and we are done. Thus we may assume that c is 0 or 2. Now $(\lambda +$ ***p***, $\alpha_2^\vee) = 3^r b \bmod 3^{r+1}$ for some $0 < \boldsymbol{b} < 3$. Hence ***b*** is 1 or 2 and we have $\boldsymbol{b} + \boldsymbol{c} \equiv 1$ mod 3 or ***c*-*br* *I*** mod3.

Subcase (a). ***b*** + c = 1 mod 3. We leave it to the reader to check that in this case

$$\lambda \to \lambda - b3^r\alpha_2 \to \lambda - b3^r\alpha_2 - 3^{r+1}\alpha$$

and

$$\lambda - 3^{r+1}\alpha \to \lambda - 3^{r+1}\alpha - b3^r\alpha_2$$

so that

$$d(\lambda, \lambda - 3^{r+1}\alpha) \leqslant \boldsymbol{3} < (2p)^2 = \boldsymbol{36.}$$

Subcase (b). ***b*** = 1, c = 2. It is not difficult to check that

$$\lambda \to \lambda - 2\ 3^{r+1}\alpha \to \lambda - 2 \cdot 3^{r+1}\alpha - 3^r\alpha_2$$

and

$$\begin{aligned}\lambda - 3^{r+1}\alpha \to \lambda - 3^{r+1}\alpha - 3^r\alpha_2 &\to \lambda - 3^{r+1}\alpha - 3^r\alpha_2 - 3^{r+1}\alpha\\ &= \lambda - 2 \cdot 3^{r+1}\alpha - 3^r\alpha_2.\end{aligned}$$

This shows that

$$d(\lambda, \lambda - 3^{r+1}\alpha) \leqslant \boldsymbol{4} < (2p)^2 = \boldsymbol{36.}$$

Subcase (c). ***b*** = 2, c = 0. Once more we give the reader a chore, this time of checking that $\lambda \to \lambda - 2 \cdot 3^r\alpha_2 \to \lambda - 2\ 3^r\alpha_2 - 2\ 3^{r+1}\alpha$ and that $\lambda - 3^{r+1}\alpha \to \lambda - 3^{r+1}\alpha - 2\ 3^r\alpha_2$. Hence, by the triangle inequality,

$$d(\lambda, \lambda - 3^{r+ \text{‘}a}) \leqslant \boldsymbol{3} + d(\lambda', \lambda' - 3^{r+\text{‘}a}), \tag{8}$$

where $\lambda' = \lambda - 3^{r+1}\alpha - 2\ 3^r\alpha_2$. **Now**

$$\begin{aligned}(\lambda' + \boldsymbol{p}, \boldsymbol{a''}) &= (\lambda + \boldsymbol{p}, \boldsymbol{a''}) - 3^{r+\text{‘}}(a, \alpha^\vee) - 2\ 3^r(\alpha_2, \alpha^\vee)\\ &= (\lambda + \rho, \alpha^\vee) - 2.3^{r+1} + 2 \cdot 3^{r+1} \equiv 0 \bmod 3^{r+2}\end{aligned}$$

and

$$(\lambda' + \rho, \alpha_2^\nu) = (\lambda + \rho, \alpha_2^\nu) - 3^{r+1}(\alpha, \alpha_2^\nu) - 4\ 3^r$$
$$= (\lambda + \rho, \alpha_2^\nu) + 3^{r+1} - 4 \cdot 3^r$$
$$= (\lambda + \rho, \alpha_2^\nu) - 3^r \equiv 3^r \bmod 3^{r+1}.$$

Thus, by Subcase (a),

$$d(\lambda', I' - 3^{r+}‘a) \leqslant 3. \tag{9}$$

Now (8) and (9) give

$$d(\lambda, \lambda - 3^{r+1}\alpha) \leqslant 6 < (2p)^2 = 36$$

as required.

This exhausts all possibilities and so completes the proof of the proposition.

Let l denote the number of simple roots.

COROLLARY 4.2. *For any positive integer r, any $\lambda \in X^{(r)}$ and any $a \in B$,*

$$d(\lambda, \lambda - p^{r+}‘a) < (2p)^l.$$

For any $\theta = \sum_{\alpha \in B} m_\alpha \alpha$ in the root lattice $\mathbb{Z}R$ we define $ht(\theta)$, the height of θ, to be $\sum_{\alpha \in B} |m_\alpha|$.

By induction on the height we obtain from Corollary 4.2.

COROLLARY 4.3. *For any positive integer r, any $\lambda \in X^{(r)}$ and any $\theta \in \mathbb{Z}R$*

$$d(\lambda, \lambda - p^{r+1}\theta) \leqslant ht(\theta)(2p)^l.$$

5. A DESCRIPTION OF THE BLOCKS

We are able to use the material of Sections 3 and 4 to build up a picture of the blocks via the following.

LEMMA 5.1. *Let λ be a dominant weight, a a simple root, $(\lambda + \rho, a”) = bp” + ap^{m+1}$, for nonnegative integers a, b, m with $0 \leqslant b < p$ and $\mu = \lambda - bp’a$. If μ is dominant it is in the same block as λ.*

Proof: Let w^+ be a nonzero element of weight λ in the Weyl module $W(\lambda)$. Now $W(\lambda)$ is the K-span of elements of the form

$$\prod_{\beta > 0} X_{-\beta, n_\beta} w^+$$

and the weight of such an element is $\lambda - \sum_{\beta > 0} n_\beta \beta$. If such an element has weight μ we must have

$$bp^m \alpha = \sum_{\beta > 0} n_\beta \beta.$$

Since α is simple this gives $n_\alpha = bp^m$ and $n_\beta = 0$ if $\beta \neq \alpha$. Hence the μ weight space of $W(\lambda)$ is spanned by $X_{-\alpha, bp^m} w^+$. Now the character of $W(\lambda)$ is given by Weyl's character formula so it follows from the proposition of Section 21.3 of [7] that μ is a weight of $W(\lambda)$. Hence $X_{-\alpha, bp^m} w^+$ is not zero.

If β is a simple root, different from α, then for any positive integer s, $X_{\beta, s}$ and $X_{-\alpha, bp^m}$ commute. Thus $X_{\beta, s}$ annihilates $X_{-\alpha, bp^m} w^+$. One may see from the lemma of Section 26.2 of [7] that $X_{\alpha, s}$ also annihilates $X_{-\alpha, bp^m} w^+$. It follows that $X_{-\alpha, bp^m} w^+$ generates a submodule of $W(\lambda)$ of highest weight μ. Thus $L(\mu)$ is a composition factor of the indecomposable module $W(\lambda)$, proving that λ and μ are in the same block.

COROLLARY 5.2. *If for $\lambda, \mu \in X^+$ there is a chain $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n = \mu$ with all $\lambda_i \in X^+$ and, for each $1 \leqslant i < n$, $\lambda_i \to \lambda_{i+1}$ or $\lambda_{i+1} \to \lambda_i$, then λ and μ lie in the same block.*

To use this corollary we shall need some very easy results.

LEMMA 5.3. *Suppose that A is a real number, m an integer, $\lambda \in X(A)$ and $\mu = \lambda + m\alpha$, for some $\alpha \in B$; then $\mu \in X(A - 3\,|m|)$.*

Proof We have

$$(\mu, \beta^v) = (\lambda, \beta^v) + m(\alpha, \beta^v) \geqslant A - 3\,|m|$$

for any $\beta \in B$, as the modulus of a Cartan integer is at most 3.

LEMMA 5.4. *Let A be a real number, r a nonnegative integer and λ, $\mu \in X^{(r)}$. If $\lambda \in X(A)$ and $d(\lambda, \mu) < \infty$ then $\mu \in X(A - 3p^{r+2} d(\lambda, \mu))$.*

Proof: By induction it suffices to consider the case in which $d(\lambda, \mu) = 1$. Thus, for some $\alpha \in B$, we have $\mu = \lambda \pm bp^s \alpha$, where $s = r$ or $r + 1$ and $0 < b < p$. Hence, by Lemma 5.3, $\mu \in X(A - 3bp^s) \subseteq X(A - 3p^{r+2})$.

PROPOSITION 5.5. *Let A be a real number, r a nonnegative integer and $\lambda, \mu \in X(A) \cap X^{(r)}$. If $A \geqslant \frac{3}{2} p^{r+2} d(\lambda, \mu)$ then λ and μ lie in the same block.*

Proof We choose a sequence $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_m = \mu$, where $m = d(\lambda, \mu)$ and, for each $1 \leqslant i < m$, $\lambda_i \to \lambda_{i+1}$ or $\lambda_{i+1} \to \lambda_i$. Then, for any i, we have $d(\lambda, \lambda_i) \leqslant m/2$ or $d(\lambda_i, \mu) \leqslant m/2$. Hence, by Lemma 5.4, $\lambda_i \in$

$X(A - \frac{3}{2}p^{r+2}m) \subseteq X(0) = X^+$. Now λ and μ are in the same block by Corollary 5.2.

For a nonnegative integer ***r, we*** define A, to be $\frac{3}{2}p^{r+2}(2p)^l\, ht(2\rho)$. The importance of the constants A, lies in the following result.

PROPOSITION 5.6. ***Let*** r ***be a nonnegative integer and*** $\lambda \in X(A_r) \cap X^{(r)}$. ***Then***

(i) $\lambda + 2mp^{r+1}\rho \in \mathscr{B}(\lambda)$ ***for any nonnegative integer m;***

(ii) ***for any*** $\theta \in \mathbb{Z}R$ ***such that*** $\lambda + p^{r+1}\theta \in X(A_r)$ ***we have A*** $+ p^{r+1}\theta \in \mathscr{B}(\lambda)$.

Proof: (i) Consider the case in which $m = 1$. By Corollary 4.3, $d(\lambda + 2p^{r+1}\rho, \lambda) \leqslant (2p)^l\, ht(2\rho)$. Hence

$$A_r \geqslant \tfrac{3}{2}p^{r+2}d(\lambda + 2p^{r+1}\rho, \rho)$$

and so $\lambda + 2p^{r+1}\rho \in \mathscr{B}(\lambda)$ by Proposition 5.5.

The general case follows by induction on ***m.***

(ii) For any nonnegative integer, ***m we*** have

$$\lambda + 2mp^{r+1}\rho, \lambda + p^{r+1}\theta + 2mp^{r+1}\rho \in X(A_r + 2mp^{r+1}). \tag{1}$$

Let ***m*** be so large that

$$A_r + 2mp^{r+1} \geqslant \tfrac{3}{2}p^{r+2}ht(\theta)(2p)^l. \tag{2}$$

By Corollary 4.3 and (2),

$$\tfrac{3}{2}p^{r+2}d(\lambda + 2mp^{r+1}\rho + p^{r+1}\theta, \lambda + 2mp^{r+1}\rho) \leqslant A_r + 2mp^{r+1}.$$

Hence, by (1) and Proposition 5.5,

$$\lambda + 2mp^{r+1}\rho + p^{r+1}\theta \in \mathscr{B}(\lambda + 2mp^{r+1}\rho). \tag{3}$$

From part (i) we obtain $\mathscr{B}(\lambda) = \mathscr{B}(\lambda + 2mp^{r+1}\rho)$ and $\mathscr{B}(\lambda + p^{r+1}\theta) = \mathscr{B}(\lambda + p^{r+1}\theta + 2mp^{r+1}\rho)$, so (3) gives $\mathscr{B}(\lambda + p^{r+1}\theta) = \mathscr{B}(\lambda)$, as required.

PROPOSITION ***5.7. Let r be a nonnegative integer and*** $\lambda \in X(A_r) \cap X^{(r)}$. ***Then***

$$(W \cdot \lambda + p^{r+1}\mathbb{Z}R) \cap X(A_r) \subseteq \mathscr{B}(\lambda).$$

Proof. We must show that, for any element w of W and $\lambda \in X(A_r) \cap X^{(r)}$,

$$(w \cdot \lambda + p^{r+1}\mathbb{Z}R) \cap X(A_r) \subseteq \mathscr{B}(\lambda). \tag{1}$$

If $w = 1$ then we obtain (1) from Proposition 5.6(ii).

Now suppose that, for some $\alpha \in B$, $w = s_\alpha$, the reflection corresponding to α. Since $\lambda \in X^{(r)}$ we have

$$(\lambda + \rho, \alpha^\vee) = bp^r + ap^{r+1}$$

for integers a and b with $0 \leqslant b < p$. For any m,

$$(\lambda + 2mp^{r+1}\rho + \rho, \alpha^\vee) = bp^r + (a + 2m)p^{r+1}$$

and $\lambda + 2mp^{r+1}\rho \in X(A_r + 2mp^{r+1})$. Thus, if m is a large positive integer

$$\lambda + 2mp^{r+1}\rho - bp^r\alpha \in X(A_r) \subseteq X^+.$$

By Lemma 5.1 and Proposition 5.6(i),

$$\lambda + 2mp^{r+1}\rho - bp^r\alpha \in \mathscr{B}(\lambda + 2mp^{r+1}\rho) = \mathscr{B}(\lambda). \tag{2}$$

Now by (1) with $w = 1$ and λ replaced by $\lambda + 2mp^{r+1}\rho - bp^{r+1}\alpha$,

$$(\lambda - bp^r\alpha + p^{r+1}\mathbb{Z}R) \cap X(A_r) \subseteq \mathscr{B}(\lambda + 2mp^{r+1}\rho - bp^r\alpha). \tag{3}$$

However,

$$s_\alpha \cdot \lambda = \lambda - (\lambda + \rho, \alpha^\vee)\alpha = \lambda - bp^r\alpha - ap^{r+1}\alpha$$

so

$$(s_\alpha \cdot \lambda + p^{r+1}\mathbb{Z}R) \cap X(A_r) = (\lambda - bp^r\alpha + p^{r+1}\mathbb{Z}R) \cap X(A_r). \tag{4}$$

Combining (2), (3) and (4) we obtain (1) with $w = s_\alpha$.

Equation (1) now follows in general by using induction on the length function on W.

THEOREM 5.8. *For any dominant weight λ,*

$$\mathscr{B}(\lambda) = (W \cdot \lambda + p^{r(\lambda)+1}\mathbb{Z}R) \cap X^+.$$

Proof. Replacing λ by an element of $(W \cdot \lambda + p^{r(\lambda)+1}\mathbb{Z}R) \cap X(A_{r(\lambda)})$, if necessary, we may assume that $\lambda \in X(A_{r(\lambda)})$. By Corollary 2.4, $(W \cdot \lambda + p^{r(\lambda)+1}\mathbb{Z}R) \cap X^+$ is a union of blocks. Suppose, for a contradiction to the theorem, that there is a block $\mathscr{B}'$ contained in $(W \cdot \lambda + p^{r(\lambda)+1}\mathbb{Z}R) \cap X^+$ such that λ is not an element of $\mathscr{B}'$. By

Proposition 5.7, $\mathscr{B}' \cap X(A_{r(\lambda)})$ is empty. However, this contradict Proposition 3.2. Thus no such $\mathscr{B}'$ exists and $\mathscr{B}(\lambda) = (W \cdot \lambda + p^{r(\lambda)+1}\mathbb{Z}R) \cap X^+$.

Remark. For any positive integer n, the affine Weyl group W_{p^n} is defined to be the group of permutations of X generated by he Weyl group and translations by elements of the form $p^n\theta$, for θ in ZR. The "dot" action of W_{p^n} on X is given by $\sigma \mu = \sigma(\mu + \rho) - \rho$, for $\sigma \in W_{p^n}$, $\mu \in$ X. It now follows from Theorem 5.8 that dominant weights λ and μ are in the same block if and only if $r(\lambda) = r(\mu)$ and λ and μ are conjugate under the dot action of $W_{p^{r(\lambda)+1}}$.

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